ABBOTSLEIGH

August 2002
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.

Total marks (84)
- Attempt Questions 1-7.
- All questions are of equal value.

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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

QUESTION 1 (12 Marks) Use a SEPARATE writing booklet. **Marks**

(a) Differentiate

(i) $\log_e\left(3x^2+2\right)$ **(1)**

(ii) $\left(1+x^2\right)\tan^{-1}x$. **(2)**

(b) Solve the inequality $\dfrac{2x}{x-2}\le 3$ **(3)**

(c) Evaluate exactly $\displaystyle\int_1^{\sqrt{3}}\frac{dt}{\sqrt{4-t^2}}$ **(2)**

(d) Using the substitution $u=4-x$ evaluate $\displaystyle\int_3^4 x\sqrt{4-x}\,dx$. **(4)**

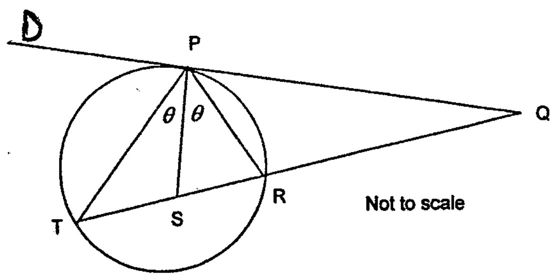
QUESTION 2 (12 Marks) Use a SEPARATE writing booklet.

(a) Evaluate $\int_0^{\pi} \cos^2 x \, dx$ **(3)**

(b) Show that $x+1$ is a factor of x^3-4x^2+x+6.
Hence or otherwise, factorise x^3-4x^2+x+6 fully. **(3)**

(c) The equation $x^3+2x-8=0$ has a root close to $x=1.6$. Use one application of Newton's method to find a better approximation to the root. (Give your answer to 2 decimal places). **(3)**

(d)



In the diagram the vertices of triangle PTR lie on a circle. The tangent at P meets the secant TR produced at Q. The bisector of $\angle TPR$ meets TR at S.

Copy the diagram into your booklet.
Prove that $PQ = SQ$. **(3)**

QUESTION 3 (12 Marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) State the domain and range of $y = 3\cos^{-1} 2x$ **(2)**

(ii) Find the value of y if $x = \frac{1}{4}$ **(1)**

(iii) Sketch the graph of $y = 3\cos^{-1} 2x$. **(1)**

(b) Let α, β, γ be the roots of the polynomial $3x^3 - 12x^2 - 8 = 0$.
Evaluate $\alpha\beta\gamma$. **(2)**

(c) If $\sin A = \frac{2}{3}$ and $\frac{\pi}{2} < A < \pi$, find the exact value of $\sin 2A$ **(2)**

(d) The acceleration of a particle x metres from 0 at time t seconds is given by

$$\frac{d^2x}{dt^2} = -e^{-2x}$$

If the velocity is 1 metre per second when $x = 0$, find the exact velocity when $x = 4$ metres. **(4)**

QUESTION 4 (12 Marks) Use a SEPARATE writing booklet.

Marks

(a) Solve $\sqrt{3}\cos x + \sin x = 1$ *for* $0 \le x \le 2\pi$. **(4)**

(b) (i) Explain why the function $f(x) = \sqrt{x-2}$ has an inverse function $f^{-1}(x)$. **(1)**

(ii) Write down the equation of the inverse function $f^{-1}(x)$ and sketch both $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes. **(3)**

(c) (i) Express $\sin A$ and $\cos A$ in terms of t where $t = \tan\dfrac{A}{2}$. **(1)**

(ii) Hence or otherwise prove that $\dfrac{\sin 2A}{1+\cos 2A} = \tan A$. **(3)**

QUESTION 5 (12 Marks) Use a SEPARATE writing booklet.

Marks

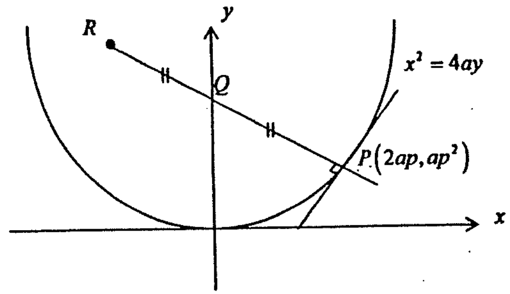
(a) Given that $f(x) = \dfrac{x}{4-x^2}$

(i) Determine whether $f(x)$ is odd, even or neither. **(1)**

(ii) Show that $f(x)$ has no stationary points. **(3)**

(iii) Find any horizontal or vertical asymptotes. **(2)**

(b)



The normal at $P(2ap, ap^2)$ on the parabola $x^2 = 4ay$ cuts the y-axis at Q and is produced to a point R such that $PQ = QR$.

(i) Given that the equation of the normal at P is $x + py = 2ap + ap^3$, find the coordinates of Q. **(1)**

(ii) Show that R has coordinates $(-2ap, ap^2 + 4a)$. **(2)**

(iii) Show that the locus of R is a parabola and state its vertex. **(3)**

QUESTION 6 (12 Marks) Use a SEPARATE writing booklet. **Marks**

(a) A point moves along the curve $y=\frac{1}{x}$ such that the x coordinate is changing at the rate of 2 units per second. At what rate is the y coordinate decreasing when $x=5$? **(3)**

(b) Molten metal at a temperature of 1400 °C is poured into moulds to form machine parts. After 15 minutes the metal has cooled to 995°C. If the temperature after t minutes is T °C, and if the temperature of the surroundings is 35°C, then the rate of cooling is approximately given by

$$\frac{dT}{dt}=-k(T-35)$$

where k is a positive constant.

(i) Show that a solution of this equation is $T=35+Ae^{-kt}$ where A is a constant. **(1)**

(ii) Find the values of A and k. **(3)**

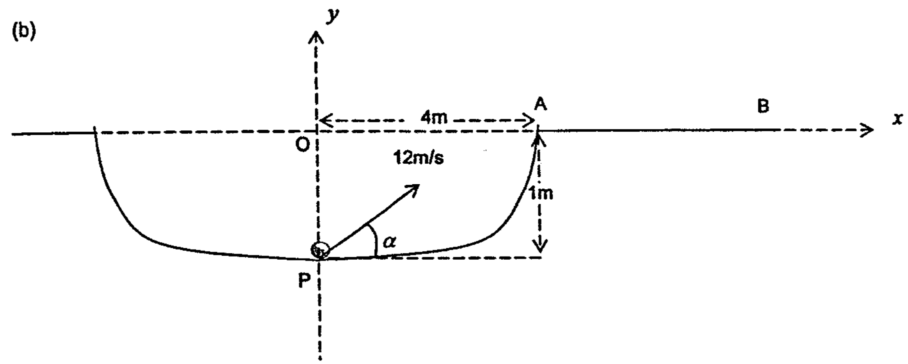
(iii) The metal can be taken out of the moulds when its temperature has dropped to 200°C. How long after the metal has been poured will this temperature be reached? **(2)**

(c) Prove by mathematical induction that $2^{3n}-3^{n}$ is divisible by 5 for all positive integers n. **(3)**

QUESTION 7 (12 Marks) Use a SEPARATE writing booklet. **Marks**

(a) Find $\lim_{x\to 0}\frac{3x}{\tan 4x}$ **(1)**

(b)



A golf ball is lying at point P, at the middle of the bottom of a sand bunker which is surrounded by level ground. The point A is at the edge of the bunker $4m$ from O and AB lies on level ground. The initial velocity is $12m/s$ and P is $1m$ below O.

(i) Using $g=-10m/s^2$, show that the golf ball's trajectory at time t seconds after being hit may be defined by the equations:

$$x=(12\cos\alpha)t \quad \text{and} \quad y=-5t^2+(12\sin\alpha)t-1$$

where x and y are the horizontal and vertical displacements, in metres, of the ball from the origin O shown in the diagram, and α is the angle of projection. **(3)**

(ii) Given $\alpha=30^\circ$, how far from A will the ball land? **(3)**

(iii) Find the range of values of α, to the nearest degree, at which the ball must be hit so that it will land to the right of A. **(4)**

END OF PAPER

Year 12 Mathematics Extension 1 Trial Solutions 2002

1) a) (i) $\frac{6x}{3x^2+2}$

(ii) $(1+x^2)\times\frac{1}{1+x^2} + \tan^{-1}x \times 2x$

$= 1 + 2x\tan^{-1}x$

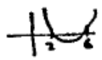
b) $\frac{2x}{x-2} \le 3 \quad x \ne 2$

$(x-2)^2 \times \frac{2x}{x-2} \le 3(x-2)^2$

$3(x-2)^2 - 2x(x-2) \ge 0$

$(x-2)(3(x-2)-2x) \ge 0$

$(x-2)(x-6) \ge 0$



$x < 2$ or $x \ge 6$

c) $\int_1^{\sqrt3} \frac{dt}{\sqrt{4-t^2}} = \left[\sin^{-1}\left(\frac{t}{2}\right)\right]_1^{\sqrt3}$

$= \sin^{-1}\frac{\sqrt3}{2} - \sin^{-1}\frac12$

$= \frac{\pi}{3} - \frac{\pi}{6}$

$= \frac{\pi}{6}$

d) $\int_3^4 x\sqrt{4-x}\,dx$

$u = 4-x$, $du = -dx$, $x=3$ $u=1$, $x=4$ $u=0$

$= \int_1^0 -(4-u)\sqrt{u}\,du$

$= \int_0^1 4u^{\frac12} - u^{\frac32}\,du$

$= \left[\frac{2\times4u^{\frac32}}{3} - \frac{2u^{\frac52}}{5}\right]_0^1$

$= \left(\frac{8\times1^{\frac32}}{3} - \frac{2\times1^{\frac52}}{5}\right) - (0-0)$

$= \frac83 - \frac25 = \frac{34}{15}$ or $2\frac{4}{15}$

2) a) $\int_0^{\pi}\cos^2x\,dx = \int_0^{\pi}\frac12(1+\cos 2x)\,dx$

$= \frac12\left[x + \frac{\sin 2x}{2}\right]_0^{\pi}$

$= \frac12\left[\left(\pi + \frac{\sin 2\pi}{2}\right) - \left(0 + \frac{\sin 0}{2}\right)\right]$

$= \frac12[\pi + 0] - 0$

$= \frac{\pi}{2}$

b) $f(x) = x^3 - 4x^2 + x + 6$

$f(-1) = (-1)^3 - 4(-1)^2 - 1 + 6$

$= -1 - 4 - 1 + 6$

$= 0$

$\therefore x+1$ is a factor of $f(x)$

$f(2) = 8 - 16 + 2 + 6$

$= 0$

$\therefore (x-2)$ is a factor

$\therefore$ Third factor must be $(x-3)$ as $1\times-2\times-3 = 6$

$f(x) = (x+1)(x-2)(x-3)$

or

$$\begin{array}{r} x^2 - 5x + 6 \\ x+1\,\overline{)\,x^3 - 4x^2 + x + 6} \\ x^3 + x^2 \\ \hline -5x^2 + x \\ -5x^2 - 5x \\ \hline 6x + 6 \\ 6x + 6 \\ \hline \end{array}$$

c) $f(x) = x^3 + 2x - 8$

$f'(x) = 3x^2 + 2$

$f(1.6) = (1.6)^3 + 2(1.6) - 8 = -0.704$

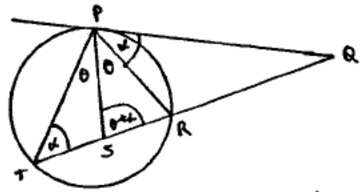
$f'(1.6) = 3(1.6)^2 + 2 = 9.68$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 1.6 - \frac{-0.704}{9.68} = 1.6727273$

$= 1.67$

d)



$\alpha = \angle QPR = \angle PTR$ (angle betw tangent & chord equals angle in alternate segment)

$\angle PSR = \alpha + \theta$ (exterior angle of Δ = sum of 2 interior opp angles)

$\angle QPS = \alpha + \theta$ (by addition)

$\therefore PQ = SQ$ (sides opp equal angles)

3 a) (i) Domain of $\cos^{-1}x$ $\quad -1 \le x \le 1$

Range of $\cos^{-1}x$ $\quad 0 \le y \le \pi$

$\frac{y}{3} = \cos^{-1}2x$

Domain $\quad -1 \le 2x \le 1$

$\therefore -\frac12 \le x \le \frac12$

Range $\quad 0 \le \frac{y}{3} \le \pi$

$0 \le y \le 3\pi$

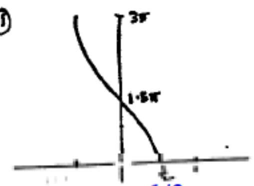
(ii) $y = 3\cos^{-1}\left(\frac{2}{4}\right)$

$= 3\cos^{-1}\left(\frac12\right)$

$= 3\times\frac{\pi}{3}$

$= \pi$

(iii)



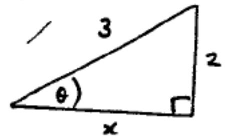
b) $\alpha\beta\gamma$ = product of roots

$3x^3 - 12x^2 + 0x - 8 = 0$

$\alpha\beta\gamma = -\frac{d}{a}$

$\alpha\beta\gamma = \frac83$

c)



By Pythagoras

$9 = 2^2 + x^2$

$x^2 = 5$

$x = \sqrt5$

$\frac{\pi}{2} < A < \pi$ 2nd quadrant

$\sin A = \frac23 \quad \cos A = -\frac{\sqrt5}{3}$

$\sin 2A = 2\sin A\cos A$

$= 2\times\frac23\times-\frac{\sqrt5}{3}$

$= -\frac{4\sqrt5}{9}$

d) $\frac{d^2x}{dt^2} = -e^{-2x}$

$acc = \frac{d}{dx}\left(\frac12 v^2\right)$

$\frac{d}{dx}\left(\frac12 v^2\right) = -e^{-2x}$

$\frac12 v^2 = \frac{+e^{-2x}}{2} + c$

When $x=0$ $v=1$

$\frac12 = \frac{e^0}{2} + c$

$\therefore c = 0$

$\frac12 v^2 = \frac{e^{-2x}}{2}$

$v^2 = e^{-2x}$

$v = \pm\sqrt{e^{-2x}}$

$v = e^{-x}$ (take +ve as $v=1$ when $x=0$)

When $x = 4$ $\quad v = e^{-4}$

$v = \frac{1}{e^4}$ metres per second

4 (a) $\sqrt3\cos x + \sin x = 1 \quad 0 \le x \le 2\pi$

Let $\sqrt3\cos x + \sin x = A\cos(x-\alpha)$

$= A\cos x\cos\alpha + A\sin x\sin\alpha$

$A\cos\alpha = \sqrt3$ (1)

$A\sin\alpha = 1$ (2)

$(1)^2 + (2)^2 \quad A^2(\sin^2\alpha + \cos^2\alpha) = (\sqrt3)^2 + 1^2$

$A^2 = 4$

$A = \pm2$ take positive $A = 2$

$(2)\div(1) \quad \tan\alpha = \frac{1}{\sqrt3}$ First quadrant as $\sin\alpha > 0$

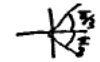
$\alpha = \frac{\pi}{6}$

$2\cos\left(x - \frac{\pi}{6}\right) = 1 \qquad 0 \le x \le 2\pi$

$0 - \frac{\pi}{6} \le x - \frac{\pi}{6} \le 2\pi - \frac{\pi}{6}$

$\cos\left(x - \frac{\pi}{6}\right) = \frac12 \qquad -\frac{\pi}{6} \le x - \frac{\pi}{6} \le \frac{11\pi}{6}$

$x - \frac{\pi}{6} = \frac{\pi}{3}$ or $2\pi - \frac{\pi}{3}$



Basic angle $= \frac{\pi}{3}$

$x - \frac{\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3}$

$x = \frac{\pi}{3} + \frac{\pi}{6}, \frac{5\pi}{3} + \frac{\pi}{6}$

$= \frac{3\pi}{6}, \frac{11\pi}{6}$

$x = \frac{\pi}{2}, \frac{11\pi}{6}$

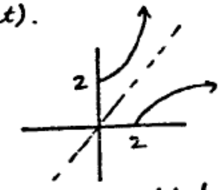
b) (i) $y = \sqrt{x-2}$ has an inverse function because it is a one-to-one function. (horizontal line test).

(ii) $x = \sqrt{y-2}$

$x^2 = y - 2$

$y = x^2 + 2$

$f^{-1}(x) = x^2 + 2$ for $x \ge 0$



Inverse function $f^{-1}(x)$ is restricted to $x \ge 0$ since it it is only half the para

c)(i) $t = \tan\frac{A}{2} \quad \sin A = \frac{2t}{1+t^2} \quad \cos A = \frac{1-t^2}{1+t^2}$

(ii) $\frac{\sin 2A}{1+\cos 2A} = \tan A$

Let $t = \tan A$ from above

$\sin 2A = \frac{2t}{1+t^2} \quad \cos 2A = \frac{1-t^2}{1+t^2}$

LHS $= \frac{2t}{1+t^2} \div \left(1 + \frac{1-t^2}{1+t^2}\right)$

$= \frac{2t}{1+t^2} \div \left(\frac{1+t^2+1-t^2}{1+t^2}\right)$

$= \frac{2t}{1+t^2} \times \frac{1+t^2}{2}$

$= t$

$= \tan A$

$=$ RHS

5 a) $f(x) = \frac{x}{4-x^2}$

(i) $f(-x) = \frac{-x}{4-(-x)^2} = \frac{-x}{4-x^2} = -f(x)$

$\therefore$ odd function

(ii) $f'(x) = \frac{(4-x^2)1 - x(-2x)}{(4-x^2)^2}$

$= \frac{4 - x^2 + 2x^2}{(4-x^2)^2}$

$= \frac{4+x^2}{(4-x^2)^2}$

$\frac{4+x^2}{...} \neq 0$ since $4+x^2>0$ for all values of x (since x^2 is always positive)

$\therefore$ since $f'(x) \neq 0$ there are no stat pts.

(ii) As $x \to \infty$ $\lim_{x\to\infty} \frac{x}{4-x^2}$

$= \lim_{x\to\infty} \frac{\frac{x}{x^2}}{\frac{4}{x^2}-1} = \frac{\frac{1}{x}}{\frac{4}{x^2}-1}$

$= \frac{0}{-1} = 0$

$y=0$ is an asymptote (horizontal)

Denominator $4-x^2 \neq 0$

$(2-x)(2+x)=0$

$x=2,\ x=2$ are asymptotes (vertical)

b)(i) $x+py = 2ap+ap^3$

sub in $x=0$ $\quad py = 2ap+ap^3$

$y = 2a+ap^2$

$Q\ (0,\ 2a+ap^2)$

(ii) $R(x,y)$ $\quad Q(0, 2a+ap^2)$ $\quad P(2ap, ap^3)$

Q is midpt so

$\frac{x+2ap}{2} = 0$ and $\frac{y+ap^2}{2} = 2a+ap^2$

$x+2ap=0$ $\qquad y+ap^2 = 4a+2ap^2$

$x=-2ap$ $\qquad y = 4a+ap^2$

$\therefore R$ is $(-2ap,\ 4a+ap^2)$

(iii) $x=-2ap$ $\quad y=4a+ap^2$

$p = \frac{-x}{2a}$ sub into y

$y = 4a + a\left(-\frac{x}{2a}\right)^2 = 4a + \frac{x^2}{4a^2}$

$y = \frac{16a^3+ax^2}{4a^2}$

$y = \frac{16a^2+x^2}{4a}$ $\qquad 4ay = x^2+16a^2$

$x^2 = 4a(y-4a)$

parabola vertex $(0, 4a)$.

5)(a) $\frac{dx}{dt} = 2$ $\qquad y = \frac{1}{x} = x^{-1}$

$\frac{dy}{dx} = -x^{-2} = -\frac{1}{x^2}$

$\frac{dy}{dt} = \frac{dx}{dt} \times \frac{dy}{dx}$

$= 2 \times -\frac{1}{x^2}$

When $x=5$

$\frac{dy}{dx} = 2 \times -\frac{1}{25}$

$= \frac{-2}{25}$

b)(i) $T = 35 + Ae^{-kt}$ (1)

$\frac{dT}{dt} = -kAe^{-kt}$

$= -k(T-35)$ from (1)

$\therefore T = 35+Ae^{-kt}$ is a soln to $\frac{dT}{dt} = -k(T-35)$

(ii) $t=0$ $T=1400$

$t=15$ $T=995$

When $t=0$ $\quad 1400 = 35 + Ae^0$

$A = 1365$

$t=15$ $\quad 995 = 35 + 1365e^{-15k}$

$\frac{960}{1365} = e^{-15k}$

$\log_e\left(\frac{960}{1365}\right) = 15k$

$k = -\frac{1}{15}\log_e\left(\frac{64}{91}\right)$

$k = 0.023465094$

(iii) $T=200$ $\quad t=?$

$200 = 35 + 1365e^{-0.023465t}$

$\frac{165}{1365} = e^{-0.023465t}$

$t = \frac{\ln\left(\frac{165}{1365}\right)}{0.023465..} = 9.04712$

It will take 90 minutes

c) $2^{3n} - 3^n$ is divisible by 5

Prove true for $n=1$

$2^3 - 3^1 = 8-3 = 5$

$\therefore$ True for $n=1$

Let it be true for $n=k$

$2^{3k} - 3^k = 5m$ where m is a positive integer

$\therefore 2^{3k} = 5m + 3^k$

Prove true for $n=k+1$

$2^{3(k+1)} - 3^{k+1} = 2^{3k} \times 2^3 - 3^{k+1}$

$= (5m+3^k)\times 8 - 3^k \times 3^1$

$= 40m + 8\times 3^k - 3\times 3^k$

$= 40m + 5\times 3^k$

$= 5(8m+3^k)$ which is divisible by 5 if m is a positive integer.

If it is true for $n=k$ we have proven it true for $n=k+1$. Since it is true for $n=1$, then it is true for $n=1+1=2$ and so on for all positive integral n.

a) $\lim_{x\to 0} \frac{3x}{\tan 4x} = \frac{3}{4}\lim_{x\to 0}\frac{4x}{\tan 4x}$

$= \frac{3}{4}\times 1$

$= \frac{3}{4}$

b)(i) $\ddot{x} = 0$

$\dot{x} = C_1$

When $t=0$ $\dot{x} = V\cos\alpha = 12\cos\alpha$

$\therefore C_1 = 12\cos\alpha$

$\dot{x} = 12\cos\alpha$

$x = 12t\cos\alpha + C_2$

When $t=0$ $x=0$ $\therefore C_2 = 0$

$x = (12\cos\alpha)t$

$\ddot{y} = -10$

$\dot{y} = -10t + C_3$

When $t=0$ $\dot{y} = V\sin\alpha = 12\sin\alpha$

$\therefore C_3 = 12\sin\alpha$

$y = -5t^2 + 12t\sin\alpha + C_4$

When $t=0$ $y=-1$ (starts at bottom of bunker).

$\therefore C_4 = -1$

$y = -5t^2 + (12\sin\alpha)t - 1$

(ii) $\alpha = 30°$. Ball will hit ground when $y=0$

$y = -5t^2 + (12\sin\alpha)t - 1$

$0 = -5t^2 + (12\sin 30°)t - 1$

$0 = -5t^2 + 6t - 1$

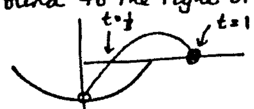
$5t^2 - 6t + 1 = 0$

$(5t-1)(t-1) = 0$

$t = \frac{1}{5}$ or $t = 1$

$t = \frac{1}{5}$ gives first time ball crosses x axis which is not on the ground (it is left of A).

$t=1$ gives the time the ball hits ground to the right of A



When $t=1$ $\quad x = (12\cos 30°)\times 1$

$= 12 \times \frac{\sqrt{3}}{2}$

$= 6\sqrt{3}$

But OA = 4 metres, so ball will land $(6\sqrt{3} - 4)$ metres from A

$6\sqrt{3} - 4$

(iii) For the ball to land to the right of A look at angle necessary to go through A

A is $(4,0)$

$4 = (12\cos\alpha)t$

$t = \frac{4}{12\cos\alpha} = \frac{1}{3\cos\alpha}$

sub into $y = -5t^2 + (12\sin\alpha)t - 1$ when $y=0$

$0 = -5\left(\frac{1}{9\cos^2\alpha}\right) + (12\sin\alpha)\left(\frac{1}{3\cos\alpha}\right) - 1$

$0 = -\frac{5}{9}\sec^2\alpha + \frac{4\sin\alpha}{\cos\alpha} - 1$

$0 = -5(1+\tan^2\alpha) + 36\tan\alpha - 9$ $\quad (\times 9)$

$= -5 - 5\tan^2\alpha + 36\tan\alpha - 9$

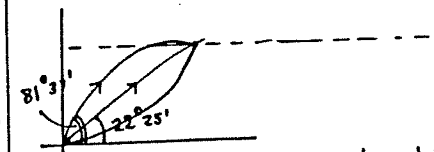
$5\tan^2\alpha - 36\tan\alpha + 14 = 0$

Use formula

$\tan\alpha = \frac{36 \pm \sqrt{36^2 - 4\times 5\times 14}}{10}$

$= 0.4125245$ or 6.78

$\alpha = 22°25'$ or $81°37'$ (first quadrant only)



Anything less than 22°25' or bigger than 81°37' will hit the bank of the bunker

So, to land to the right of A

$23° \leq \alpha \leq 81°$ (to nearest degree)